Section 5.4: Basis & Dimension

We have an idea of "dimension", but this section will make it more precise.

Lets start by looking at coordinate systems:

We can use vectors rather than coordinate axes to specify the coordinate system:

Vectors that specify a coord. system are called "basis vectors" for that system, & non-zero vectors of any length will do the job. The lengths of the basis vectors correspond to distances between successive integer points, "scale", & their directions define the positive directions of the axes.

Defn': If V is any vector space & $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a set of vectors in V, then S is called a "*basis*" for V if:

a) S is linearly independent.

b) S spans V.

Thm': Uniqueness of Basis Representation

If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for a vector space V, then every vector $\vec{v}$ in V can be expressed as $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$ in EXACTLY ONE way.

Defn': If $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis for a vector space V, and $\underbrace{\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n}_{\text{vector } \vec{v} \text{ in terms of basis } S}$,

$\Rightarrow c_1, c_2, \dots, c_n$ are "coordinates" of $\vec{v}$ relative to the basis S, & vector $(c_1, c_2, \dots, c_n)$ in $\mathbb{R}^n$ is the "coordinate vector of $\vec{v}$ relative to S".

$$(\vec{v})_S = (c_1, c_2, \dots, c_n)$$

ex/ Find a basis for $\mathbb{R}^3$.

Extending this idea...

$e_1=(1,0,\ldots,0)$, $e_2=(0,1,0,\ldots,0)$, ..., $e_n=(0,0,\ldots,0,1)$ is linearly indep. & spans $\mathbb{R}^n$ since any $\vec{v}$ in $\mathbb{R}^n$

$$\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n .$$

This is the "standard basis for $\mathbb{R}^n$".

ex// Let $\vec{v}_1=(1,2,1)$, $\vec{v}_2=(2,9,0)$, & $\vec{v}_3=(3,3,4)$. Show that $S=\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

ex// Show that $S = \{1, x, x^2, \dots, x^n\}$ is a basis for P_n.

ex// For $p_1 = 4 + 6x + x^2$, $p_2 = -1 + 4x + 2x^2$, $p_3 = 5 + 2x - x^2$, does $S = \{p_1, p_2, p_3\}$ form a basis for P_2?

Defn!: A nonzero vector space V is called "finite-dimensional" if it contains a finite set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ that forms a basis. If no such set exists, V is "infinite-dimensional".

Thm!: Let V be a finite-dimensional vectorspace, & let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be any basis:

a) If a set has more than n vectors $\Rightarrow$ it is linearly dependent.

b) If a set has fewer than n vectors $\Rightarrow$ it does not span V.

Thm!: All bases for a finite-dimensional vector space have the same number of vectors!

<u>Defn'</u>: The "dimension" of a finite dimensional vector space V, dim(V), is the number of vectors in the basis for V.

ex// What is...

a) $\dim(\mathbb{R}^n)$

b) $\dim(P_n)$

c) $\dim(M_{mn})$

ex// Determine a basis for & the dimension of the solution space of:

$$\begin{aligned} 2x_1+2x_2-x_3+\quad x_5&=0\\ -x_1-x_2+2x_3-3x_4+x_5&=0\\ x_1+x_2-2x_3-x_5&=0\\ x_3+x_4+x_5&=0 \end{aligned}$$

Thm': Plus–Minus Theorem

Let S be a non-empty set of vectors in vector space V:

a) If S is a linearly indep. set & if $\vec{v}$ is a vector in V that is outside of span (S), then the set $S \cup \{\vec{v}\}$ that results by inserting $\vec{v}$ into S is still linearly indep.

b) If $\vec{v}$ is a vector in S that is a linear combo. of other vectors in S, & if $S - \{\vec{v}\}$ denotes the set where $\vec{v}$ is removed, then

$$\text{span}(S) = \text{span}(S - \{\vec{v}\}).$$

<u>Thm</u>: If V is an n-dimensional vector space, & if S is a set in V with exactly n vectors, then S is a basis for V if <u>either</u> S spans V or S is linearly indep.

ex// a) Show that $\vec{v}_1 = (-3, 7)$ & $\vec{v}_2 = (6, 6)$ form a basis for $\mathbb{R}^2$.

b) Show that $\vec{v}_1 = (2, 0, -1)$, $\vec{v}_2 = (4, 0, 7)$, & $\vec{v}_3 = (-1, 1, 4)$ form a basis for $\mathbb{R}^3$.

<u>Thm</u>': Let S be a finite set of vectors in a finite-dim. vector space V:

a) If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from V.

b) If S is a linearly indep. set that is not a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

__Thm__: If W is a subspace of a finite-dim vector space V, then $\dim(W) \leq \dim(V)$; & if $\dim(W) = \dim(V)$, then $W = V$.

We can see this in $\mathbb{R}^3$: